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MONTE CARLO GENERATION OF THE ENERGY SPECTRUM OF SYNCHROTRON RADIATION

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Abstract

A very fast and precise algorithm suitable for the Monte Carlo generation of the standard synchrotron radiation spectrum is described. The algorithm is based on direct inversion of the cumulative distribution, using a small set of intervals, simple transformations and Chebyshev polynomials. Comparison with other algorithms, and the implementation into the program package Geant4 are also described.

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1 Introduction and standard formulas

Synchrotron radiation photons are emitted by relativistic charged particles travelling in magnetic fields. The properties of synchrotron radiation are well understood and described in textbooks [1–3].

As discussed by Schwinger [4] in his classical paper, the power radiated by an accelerated particle of charge e is described by the relativistic version of Lamor's formula

$$P = \frac{e^2 \gamma^2}{6\pi \epsilon_0 m^2 c^3} (\dot{\mathbf{p}}^2 - \beta^2 \dot{p}^2) \quad (1)$$

where $\dot{\mathbf{p}}$ and \dot{p} are the time derivatives of the particles momentum vector and absolute value, m the mass of the particle and $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$ the usual Lorentz quantities. The (Lorentz) force \mathbf{F} acting on a particle with charge e in an electromagnetic field is

$$\mathbf{F} = \dot{\mathbf{p}} = e (\mathbf{E} + \mathbf{v} \times \mathbf{B}) . \quad (2)$$

The power lost in radiation by linear acceleration in the direction of the particles is very small. Perpendicular acceleration instead as produced by magnetic fields will result in appreciable radiation for relativistic particles.

This can be seen as follows. The magnetic field only changes the direction and not the absolute value of the momentum ($\dot{p} = 0$). The term in the bracket of Eq. 1 is then simply $\dot{\mathbf{p}}^2$.

For acceleration in the direction of the motion as possible with electric fields, we have instead $\dot{\mathbf{p}}^2 = \dot{p}^2$. For the bracket in Eq. 1 we now get a cancellation by $\dot{\mathbf{p}}^2 - \beta^2 \dot{p}^2 = \dot{p}^2(1 - \beta^2) = \dot{p}^2/\gamma^2$. Radiation by linear acceleration is suppressed by γ^2 which makes it negligible for relativistic particles.

1.1 Special case of circular motion

In the simplest case, we have an electron of momentum p moving perpendicular to the homogeneous magnetic field B . The magnetic field will keep the particle on a circular path, with radius

$$\rho = \frac{p}{eB} = \frac{m\gamma\beta c}{eB} . \quad \text{Numerically we have approximately } \rho[\text{m}] = p[\text{GeV}/c] \frac{3.336 \text{ m}}{B[\text{T}]} . \quad (3)$$

We use the standard definition of the critical photon energy and frequency

$$E_c = \frac{3}{2} \frac{\hbar c \gamma^3}{\rho} \quad \omega_c = \frac{3}{2} \frac{c \gamma^3}{\rho} \quad (4)$$

in which half of the synchrotron radiation power is radiated above the critical energy.

With x we denote the photon energy E_γ , expressed in units of the critical energy E_c

$$x = \frac{E_\gamma}{E_c} . \quad (5)$$

The photon spectrum (number of photons emitted per path length s and relative energy x interval) can be written as

$$\frac{d^2 N}{ds dx} = \frac{\sqrt{3} \alpha}{2\pi} \frac{\gamma}{\rho} \int_x^\infty K_{5/3}(\xi) d\xi \quad (6)$$

where $\alpha = e^2 / 4\pi\epsilon_0 \hbar c$ is the dimensionless electromagnetic coupling (or fine structure) constant and $K_{5/3}$ a modified Bessel function of the third kind.

With

$$\frac{\gamma}{\rho} = \frac{eB}{m\beta c} \approx \frac{eB}{mc} \quad \frac{mc}{e} = 1.7045 \times 10^{-3} \text{ Tm} \quad (7)$$

we can also write the spectrum in terms of the field rather than ρ .

The number of photons emitted per length interval and the mean free path λ between two photon emissions is obtained by integration over all photon energies. Using

$$\int_0^\infty dx \int_x^\infty K_{5/3}(\xi) d\xi = \frac{5\pi}{3} \quad (8)$$

we have that

$$\frac{dN}{ds} = \frac{5\alpha}{2\sqrt{3}} \frac{\gamma}{\rho} = \frac{5\alpha}{2\sqrt{3}} \frac{eB}{m\beta c} = \frac{1}{\lambda}. \quad (9)$$

Here we are only interested in ultra-relativistic ($\beta \approx 1$) particles, for which λ only depends on the field B and not on the particle energy

$$\lambda = \frac{\lambda_B}{B} \quad \text{with} \quad \lambda_B = \frac{2\sqrt{3}}{5} \frac{mc}{\alpha e} = 0.16183 \text{ Tm}. \quad (10)$$

Before we turn to the more general case with an arbitrary angle between motion and magnetic field, some typical numbers.

A 10 GeV electron, travelling perpendicular to a 1 T field moves along a circular path of radius $\rho = 33.356 \text{ m}$. For the Lorentzfactor we have $\gamma = 19569.5$ and $\beta = 1 - 1.4 \times 10^{-9}$. The critical energy is $E_c = 66.5 \text{ keV}$ and the mean free path between two photon emissions $\lambda = 0.16183 \text{ m}$.

1.2 Generalisation to an arbitrary angle between the particle motion and the field

We now consider the general case with an arbitrary angle θ between the magnetic field \mathbf{B} and the momentum \mathbf{p} of the particle. In addition to the circular motion in the plane perpendicular to the magnetic field, there is now also a constant momentum component parallel to the magnetic field. The particle trajectory will be a helix. The generalised formulas can be derived from the special case given before by Lorentz transformation.

For a particle of momentum \mathbf{p} and magnetic field \mathbf{B} with arbitrary angle θ between these (such that $\mathbf{p}\mathbf{B} = pB \cos \theta$) we now have for the critical energy

$$E_c = \frac{3}{2} \hbar c \frac{\gamma^3 \sin \theta}{\rho} = \frac{3\hbar}{2m} \gamma^2 eB \sin \theta = \frac{3\hbar}{2m} \gamma^2 eB_\perp \quad (11)$$

and for the mean free path length along the helix

$$\lambda = \frac{\lambda_B}{B \sin \theta} = \frac{\lambda_B}{B_\perp} \quad (12)$$

and for the spectrum

$$\frac{d^2 N}{ds dx} = \frac{\sqrt{3}\alpha}{2\pi} \frac{eB_\perp}{mc} \int_x^\infty K_{5/3}(\xi) d\xi \quad (13)$$

with $B_\perp = B \sin \theta$.

The power radiated in the synchrotron radiation photons accounts (for $\beta \approx 1$) fully for the power in the general expression for the radiation of an accelerated charge as given in Eq. 1.

The photons are emitted in the direction of the moving particle in a narrow cone of about $1/\gamma$ opening.

1.3 Validity

The spectrum given in Eq. 13 can generally be expected to provide a very accurate description for the synchrotron radiation spectrum generated by GeV electrons in magnetic fields.

Here we discuss some known limitations and possible extensions.

For particles, travelling on a circular path, the spectrum observed in one location at the ring will in fact not be a continuous spectrum, but a discrete spectrum, consisting only of harmonics or modes n of the revolution frequency. In practice, the mode numbers will generally be too high to make this a visible effect. The critical mode number corresponding to the critical energy and frequency is $n_c = 3/2 \gamma^3$. For 10 GeV electrons for example we have $n_c \approx 10^{13}$.

We used $\beta = 1$ to calculate the number of photons and the energy spectrum. This introduces an uncertainty of about $1/2\gamma^2$ or less than 5×10^{-7} for $\gamma > 10^3$.

It is rather straightforward to extend the formulas presented here to other particles than electrons, with arbitrary charge q and mass m , see [5]. The number of photons and the power scales with the square of the charge.

The standard synchrotron spectrum of Eq. 13 is only valid as long as the photon energy remains small compared to the particle energy [6, 7]. This is a very safe assumption for GeV electrons and commercially available magnets with fields of the order of Tesla.

An extension of synchrotron radiation to fields exceeding several hundred Tesla as present in the beam-beam interaction in linear-colliders, is also known as beamstrahlung. For an introduction see [8].

The standard photon spectrum applies to homogenous fields and remains a good approximation for magnetic fields which remain approximately constant over a the length ρ/γ also known as formation length for synchrotron radiation. Short magnets and edge fields will result instead in more energetic photons than predicted by the standard spectrum.

We also note that short bunches of many particles will start to radiate coherently like a single particle of the equivalent charge at wavelengths which are longer than the bunch dimensions.

Low energy, long wavelength synchrotron radiation may destructively interfere with conducting surfaces [9].

The soft part of the synchrotron radiation spectrum emitted by charged particles travelling through a medium will be modified for frequencies close to and lower than the plasma frequency [10].

2 Direct inversion and generation of the photon energy spectrum

The task is to find an algorithm, that effectively transforms the flat distribution given by standard pseudo-random generators into the desired distribution proportional to the expressions given in Eqs. 13, 14.

There are standard techniques to generate an arbitrary distribution, see for instance [11] or Section 33 in [12]. Generally, to obtain the probability density function $f(x)$, the transformation can be constructed from the inverse F^{-1} of the cumulative distribution function $F(x) = \int_0^x f(t)dt$.

Leaving aside constant factors, the probability density function relevant for the photon energy spectrum is

$$\text{SynRad}(x) = \int_x^\infty K_{5/3}(t)dt . \quad (14)$$

Numerical methods to evaluate $K_{5/3}$ are discussed in [13]. An efficient algorithm to evaluate the integral SynRad using Chebyshev polynomials is described in [14]. This has been used by the

author in an earlier version of the Monte Carlo generator for synchrotron radiation using approximate transformations and the rejection method [15].

The cumulative distribution function is the integral of the probability density function. Here we have

$$\text{SynRadInt}(z) = \int_z^\infty \text{SynRad}(x) dx, \quad (15)$$

with normalization

$$\text{SynRadInt}(0) = \int_0^\infty \text{SynRad}(x) dx = \frac{5\pi}{3}, \quad (16)$$

such that $\frac{3}{5\pi}\text{SynRadInt}(x)$ gives the fraction of photons above x .

The task is to find an algorithm which accurately describes the inverse of this function. It is possible to perform the inversion numerically using a look-up-table. Such an approach was followed in [16] and [17].

It is also possible to directly obtain the desired distribution as fast and accurate algorithm using an analytical description based on simple transformations and Chebyshev polynomials. This approach is used here. We now describe in some detail how the analytical description was obtained.

It turned out to be convenient to start from the normalized complement rather than Eq.15 directly, that is

$$\text{SynFracInt}(x) = \frac{3}{5\pi} \int_0^x \int_x^\infty K_{5/3}(t) dt dx = 1 - \frac{3}{5\pi} \text{SynRadInt}(x), \quad (17)$$

which gives the fraction of photons below x .

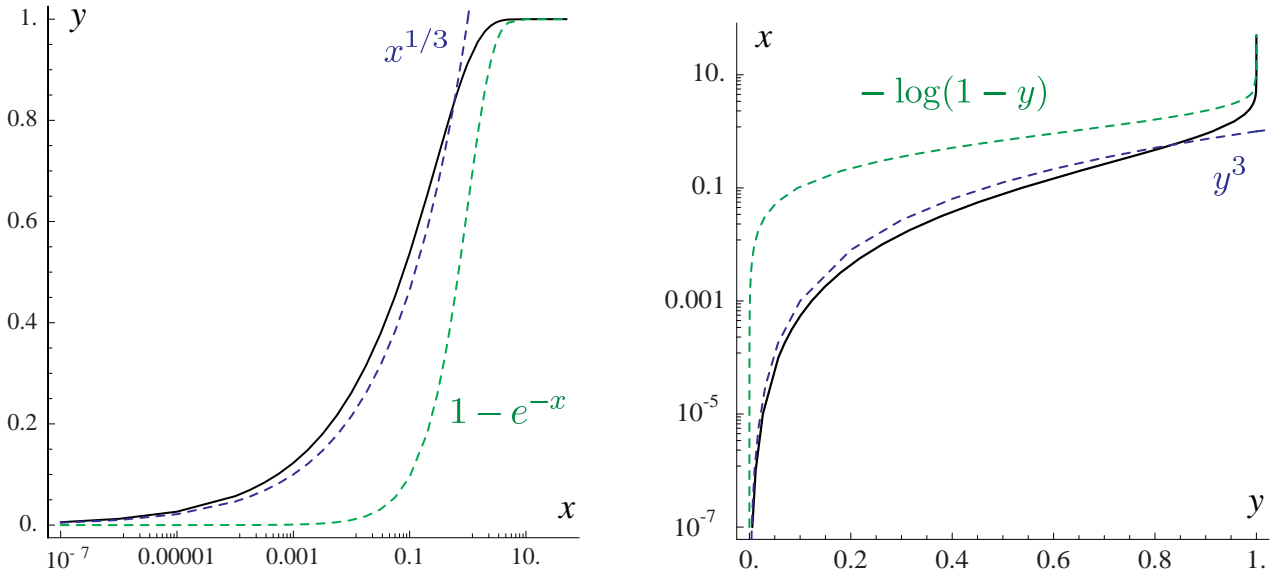


Figure 1: SynFracInt (left) and its inverse InvSynFracInt (right), on a log x scale. The functions $x^{1/3}$, y^3 and $1 - e^{-x}$, $-\log(1 - y)$, which are used as simple approximate expressions for small and large arguments are shown as dashed lines.

Figure 1 shows on the left hand side $y = \text{SynFracInt}(x)$ and on the right hand side the inverse $x = \text{InvSynFracInt}(y)$ together with simple approximate functions. Chebyshev polynomials can be directly fitted to $\text{InvSynFracInt}(y)$ for intermediate y values with good convergence. Close to 0 and 1, the function is too steep and a combination of approximate asymptotic expressions and Chebyshev polynomials is used.

The approximate expression can be derived from known asymptotic expressions for Bessel functions by integration and inversion. SynFracInt starts approximately as $x^{1/3}$ and saturates like $1 - e^{-x}$ for $x \rightarrow 1$.

For the inverse, $\text{InvSynFracInt}(y)$ which we need for the Monte Carlo generation, the approximate expressions are y^3 for small y and $-\log(1 - y)$ close to 1.

The division in three intervals with the expressions used in each case are given in Table 1. P_{Ch} stands for the Chebyshev polynomials. The convergence is good. Full (64 bit) machine precision is reached with less than 30 coefficients in each interval.

Table 1: Intervals used for InvSynFracInt .

y	$x = \text{InvSynFracInt}(y)$
$y < 0.7$	$y^3 P_{\text{Ch},1}(y)$
$0.7 \leq y \leq 0.91322603$	$P_{\text{Ch},2}(y)$
$y > 0.91322603$	$-\log(1 - y) P_{\text{Ch},3}(-\log(1 - y))$

The procedure for the Monte Carlo simulation is to generate y at random, uniformly distributed between 0 at 1, as provided by standard random generators, and then to calculate the energy x in units of the critical energy according to $x = \text{InvSynFracInt}(y)$.

Table 2: Numerical values.

y	y^3	$-\log(1 - y)$	$x = \text{InvSynFracInt}(y)$
0.0571307	0.0001865		0.0001
0.1228125	0.0018524		0.001
0.2618744	0.017959		0.01
0.5371636	0.154996		0.1
0.91322603		2.4444	1
0.9992627768		7.21262	5
0.999996506475		12.5646	10
0.999999999889		22.9173	20

Numerical values are given in Table 2. We can see that y^3 and $-\log(1 - y)$ already get us reasonable close to x , which allows for good convergence of the Chebyshev polynomials.

3 Properties of the photon energy and power spectra

The normalised probability function describing the photon energy spectrum is

$$n_\gamma(x) = \frac{3}{5\pi} \int_x^\infty K_{5/3}(t) dt. \quad (18)$$

$n_\gamma(x)$ gives the fraction of photons in the interval x to $x + dx$, where x is the photon energy in units of the critical energy.

The first moment or mean value is

$$\mu = \int_0^\infty x n_\gamma(x) dx = \frac{8}{15\sqrt{3}}. \quad (19)$$

implying that the mean photon energy is $\frac{8}{15\sqrt{3}} = 0.30792$ of the critical energy.

The second moment about the mean or variance is

$$\sigma^2 = \int_0^\infty (x - \mu)^2 n_\gamma(x) dx = \frac{211}{675}, \quad (20)$$

and the r.m.s value of the photon energy spectrum $\sigma = \sqrt{\frac{211}{675}} = 0.5591$.

The normalised power spectrum is

$$P_\gamma(x) = \frac{9\sqrt{3}}{8\pi} x \int_x^\infty K_{5/3}(t) dt . \quad (21)$$

$P_\gamma(x)$ gives the fraction of the power which is radiated in the interval x to $x + dx$.

Half of the power is radiated below the critical energy

$$\int_0^1 P_\gamma(x) dx = 0.5000 \quad (22)$$

The mean value is of the power spectrum is

$$\mu = \int_0^\infty x P_\gamma(x) dx = \frac{55}{24\sqrt{3}} = 1.32309 . \quad (23)$$

The variance is

$$\sigma^2 = \int_0^\infty (x - \mu)^2 P_\gamma(x) dx = \frac{2351}{1728} , \quad (24)$$

and the r.m.s width $\sigma = \sqrt{\frac{2351}{1728}} = 1.16642$.

4 Comparison

The numerical accuracy of the energy spectrum presented here is about 14 decimal digits, close to the machine precision.

Figure 2 shows a comparison of generated and expected spectra.

Figure 3 shows the accuracy in the energy spectrum of the Monte Carlo generator of ref. [16] which uses a table and interpolation techniques. The accuracy is of the order of 1% below the critical energy and increases to about 10% for high photon energies. This generator is implemented in the DIMAD [18], MAD8 [19] and MAD-X [20] programs.

Figure 4 shows the accuracy using the first implementation of synchrotron radiation in Geant4 [17] based on a look up table without interpolation, which resulted in large bin to bin fluctuations and no photons above $4 E_{cr}$.

5 Geant4 implementation

The synchrotron radiation spectrum generator described here has been fully implemented in GEANT4. The code and an example for its use (extended/electromagnetic/TestEm16) are provided with the standard GEANT4 distribution and available from the web [21].

The example provided with the GEANT4 code was used to produce Fig. 5. The geometry is a box of $500 \text{ m} \times 500 \text{ m} \times 500 \text{ m}$ with vacuum centered around the origin. A positron enters the box in x direction. Inside the box, there is a homogeneous magnetic field in z direction of 0.1 T. The positron is bend downwards by the magnetic field and photons are radiated tangentially.

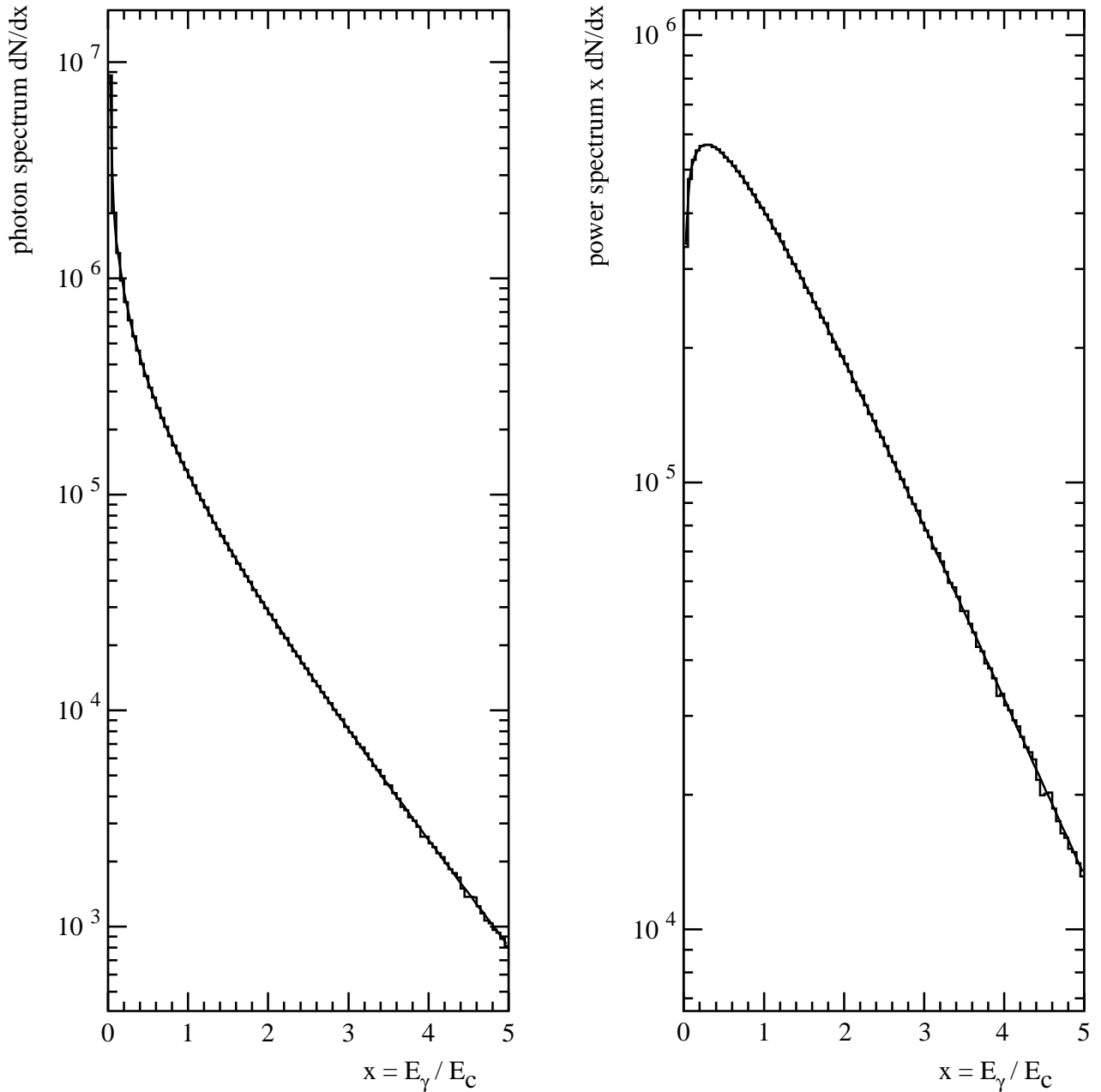


Figure 2: Comparison of the exact (smooth curve) and generated (histogram) spectra for 2×10^7 events. The photon spectrum is shown on the left and the power spectrum on the right side.

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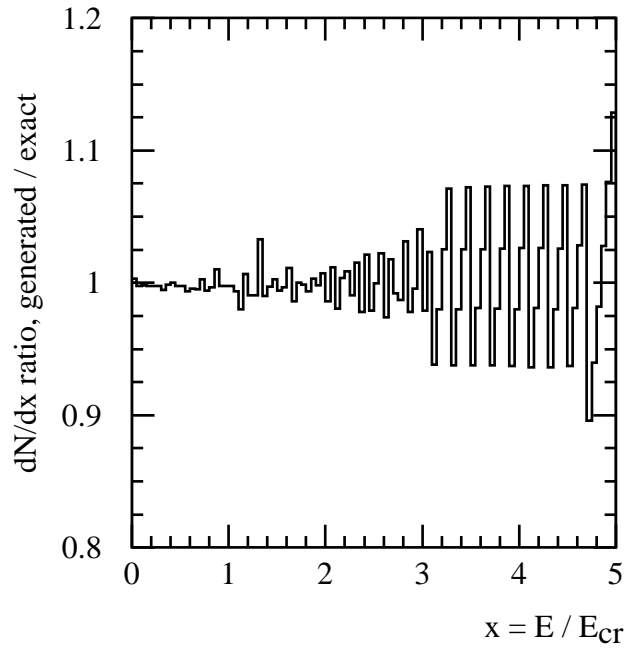


Figure 3: Accuracy of the generator by G.Roy [16].

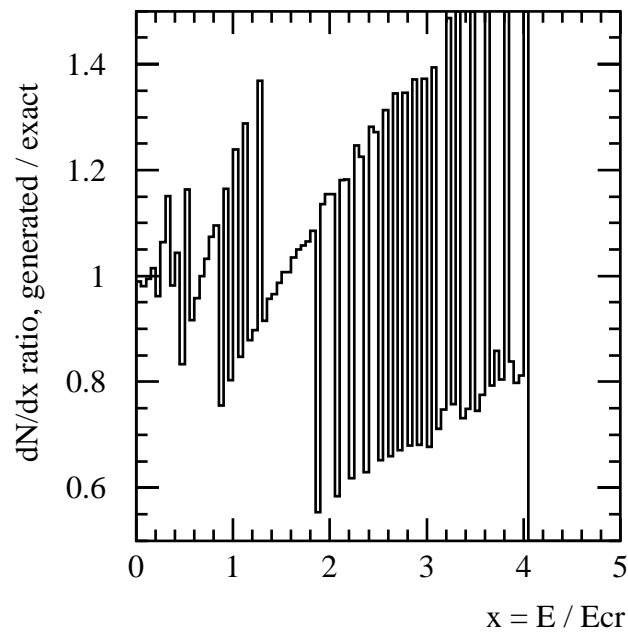


Figure 4: Accuracy of the earlier Geant4 generator (up to the version geant4-07 end of 2005). [17]

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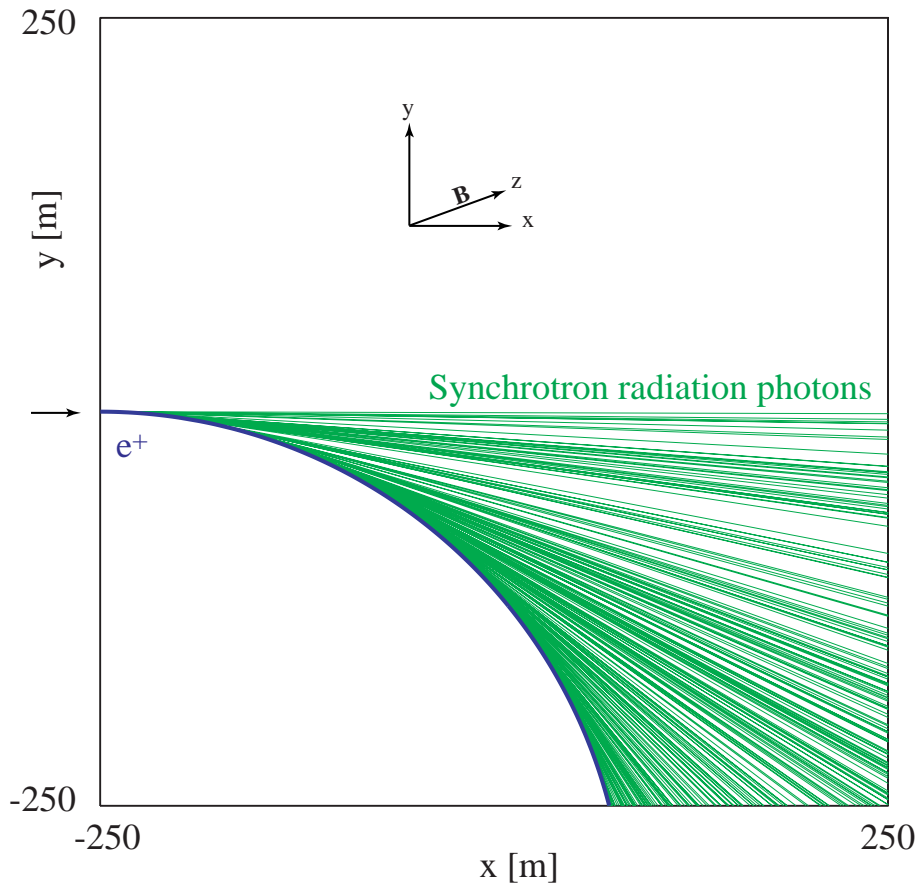


Figure 5: Geant4 display. 10 GeV e^+ moving initially in x-direction, bend downwards on a circular path by a 0.1 T magnetic field in z-direction.

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